

# Stokes' Theorem

(Mathematical Physics, B.Sc. (Physics) Part-III, Paper-V, Group-A)

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*“I never did anything by accident, nor did any of my inventions come by accident; they came by work.”*

— Stephen Hawking (1942-2018)

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In the earlier lecture notes, we have discussed divergence theorem and Green's theorem. Here we discuss Stokes' theorem which relates surface integral of a derivative of a function to the line integral of the function and the line integral is evaluated over the perimeter bounding the surface. Stokes's theorem, in equation form, is written as

$$\boxed{\oint_{\mathcal{L}} \mathbf{V} \cdot d\mathbf{l} = \iint_{\mathcal{A}} \nabla \times \mathbf{V} \cdot d\mathbf{a}}, \quad (1)$$

which states that line integral of a continuous vector function  $\mathbf{V}$  around a closed curve  $\mathcal{L}$  is equal to normal surface integral of curl  $\mathbf{V}$  over an open surface bounded by  $\mathcal{A}$  provided that first derivative of  $\mathbf{V}$  is continuous.

**Proof:** In order to prove the above theorem, we consider the right-hand side of Eq. (1) and write it in component form as

$$\iint_{\mathcal{A}} (\nabla \times \mathbf{V}) \cdot \hat{\mathbf{n}} da = \iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}}V_x + \nabla \times \hat{\mathbf{j}}V_y + \nabla \times \hat{\mathbf{k}}V_z) \cdot \hat{\mathbf{n}} da. \quad (2)$$

Next, we consider the first term on the right-hand side of above equation and write it in component form as

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}}V_x) \cdot \hat{\mathbf{n}} da = \iint_{\mathcal{A}} \left( \frac{\partial V_x}{\partial z} \hat{\mathbf{j}} \cdot \hat{\mathbf{n}} - \frac{\partial V_x}{\partial y} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \right) da. \quad (3)$$

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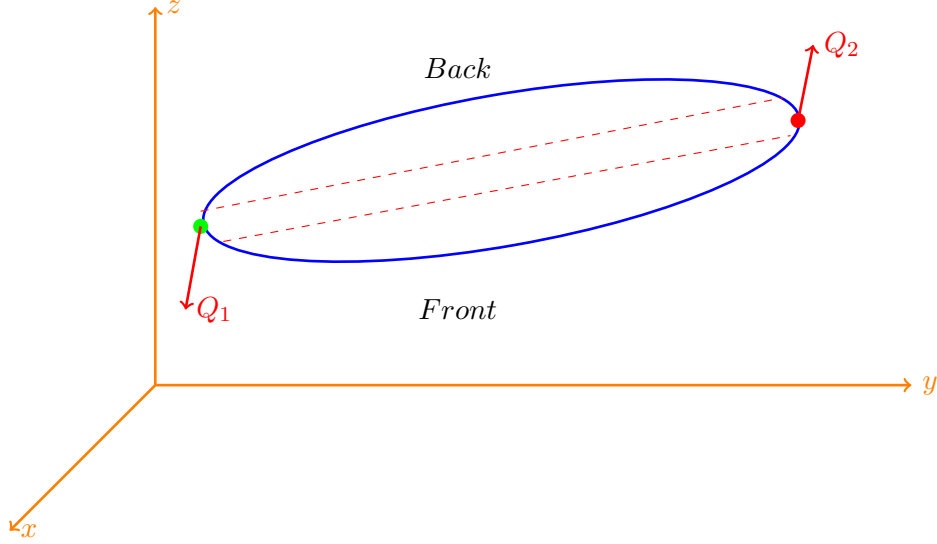


FIG. 1: (i).

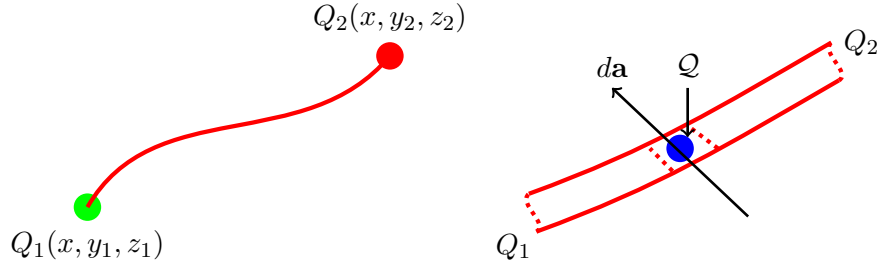


FIG. 2: (ii).

See the Figures (1) and (2), the projection of  $d\mathbf{a}$  on the  $xy$ -plane gives us the following relation

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} da = dx dy . \quad (4)$$

Next, we consider the segment  $\overline{Q_1Q_2}$  (See figures (1) and (2)) be the intersection of the surface  $\mathcal{A}$  with a plane which is parallel to the  $yz$ -plane at a distance  $x$  from the origin. Now along the segment  $\overline{Q_1Q_2}$  we can write

$$dV_x = \frac{\partial V_x}{\partial y} dy + \frac{\partial V_x}{\partial z} dz , \quad (5)$$

and tangent  $d\mathbf{r}$  (say) to the segment  $\overline{Q_1Q_2}$  at  $Q$  is given by

$$d\mathbf{r} = dy \hat{\mathbf{j}} + dz \hat{\mathbf{k}} , \quad (6)$$

which is perpendicular to  $\hat{\mathbf{n}}$ . Next, we can write

$$d\mathbf{r} \cdot \hat{\mathbf{n}} = 0 = dy \hat{\mathbf{j}} \cdot \hat{\mathbf{n}} + dz \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} , \quad (7)$$

which gives

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{n}} = -\frac{dz}{dy} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = -\frac{dz}{dy} \left( \frac{dx}{da} \right) \quad (8)$$

or

$$\hat{\mathbf{j}} \cdot \hat{\mathbf{n}} da = -dx dz . \quad (9)$$

Now using relations (4), (5) and (9) in Eq. (3) and obtain

$$\begin{aligned} \iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}}V_x) \cdot \hat{\mathbf{n}} da &= \iint_{\mathcal{A}} \left( \frac{\partial V_x}{\partial z} \hat{\mathbf{j}} \cdot \hat{\mathbf{n}} - \frac{\partial V_x}{\partial y} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \right) da \\ &= \iint_{\mathcal{A}} \left( -\frac{\partial V_x}{\partial z} dx dz - \frac{\partial V_x}{\partial y} dx dy \right) \\ &= -\iint_{\mathcal{A}} \left( \frac{\partial V_x}{\partial z} dz + \frac{\partial V_x}{\partial y} dy \right) dx \\ &= -\int dx \int dV_x \\ &= -\int [V_x(x, y_2, z_2) - V_x(x, y_1, z_1)] dx . \end{aligned} \quad (10)$$

See the figures, the periphery at  $Q_1$  is positive,  $dx = dl_x$  and it is negative at  $Q_2$ ,  $dx = -dl_x$ .

Now the above equation can be written as

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}}V_x) \cdot \hat{\mathbf{n}} da = \int V_x(x, y_2, z_2) dl_x + \int V_x(x, y_1, z_1) dl_x . \quad (11)$$

The first term on the right-hand side of above equation represents the back part and the second term represents the front part shown in the Fig. (1). The sum of both terms gives  $\int_{\mathcal{L}} V_x dl_x$ . Therefore, we can write

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{i}}V_x) \cdot \hat{\mathbf{n}} da = \oint_{\mathcal{L}} V_x dl_x . \quad (12)$$

Using similar procedure we obtain

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{j}}V_y) \cdot \hat{\mathbf{n}} da = \oint_{\mathcal{L}} V_y dl_y \quad (13)$$

and

$$\iint_{\mathcal{A}} (\nabla \times \hat{\mathbf{k}}V_z) \cdot \hat{\mathbf{n}} da = \oint_{\mathcal{L}} V_z dl_z . \quad (14)$$

Now combining Eqs. (12), (13) and (14), we obtain

$$\boxed{\iint_{\mathcal{A}} (\nabla \times \mathbf{V}) \cdot \hat{\mathbf{n}} \, da = \oint_{\mathcal{L}} \mathbf{V} \cdot d\mathbf{l}} . \quad (15)$$

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